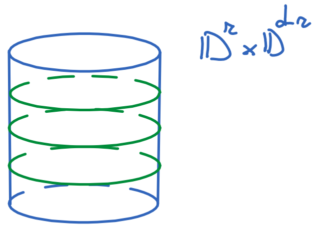
2 - Foliations

giovedì 26 ottobre 2023 16:21

Regular foliations

X regular manifold of dimension d
(regular) locally looks like an open subset of $\mathbb{C}^d$.

A (regular) foliation $\mathcal{F}$ on X (of dimension r)
locally looks like the decomposition of $\mathbb{D}^d$ into

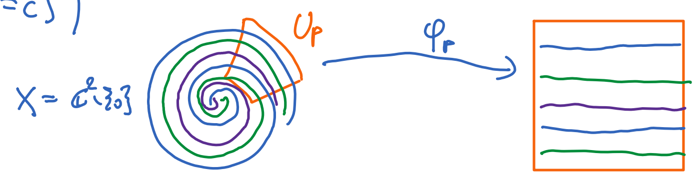


$\mathbb{D}^d = \bigsqcup_{y \in \mathbb{D}^q} \mathbb{D}^r \times \{y\}$ $\qquad (q = d-r)$
↖ standard foliation, denoted $\mathcal{F}_{st}^{[r,q]}$.

The sets $\mathbb{D}^r \times \{y\}$ are called plaques (or local leaves).

We formalise the definition of regular foliation in three equivalent ways:

- by local model: a foliation $\mathcal{F}$ on X is a decomposition $X = \bigsqcup_{\omega \in \Omega} L_\omega$ of X into immersed submanifolds L_ω of dimension r, such that $\forall p \in X$ $\exists U_p \ni p$ open set and $\varphi_p : U_p \to V_p \underset{(\subseteq)}{=} \mathbb{D}^d$ biholomorphism such that $L_\omega \cap U_p = \varphi_p^{-1}\left(\bigsqcup_{c \in S_\omega} \{y = c\} \right)$



- by foliation charts: $\mathcal{F}$ is (the equivalence class of) an atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ of X, with $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{D}^d = \mathbb{D}^r \times \mathbb{D}^q$ such that the transition maps are skew-products: $\psi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$

$$\psi_{\alpha\beta}(x,y) = \big(f(x,y),\ g(y) \big) \qquad \text{↖ biholomorphism}$$

With this description, the plaques $\varphi_\alpha^{-1}(y = c)$ glue together to give the leaves of the foliation $\mathcal{F}$.

Rem: more precisely, the leaves of $\mathcal{F}$ are equivalence classes for the equivalence relation $\sim$ on X generated by $\varphi_\alpha^{-1}(x, y_0) \sim \varphi_\alpha^{-1}(x', y_0)$ $\forall \alpha$, $\forall y_0 \in \mathbb{D}^q$ $\forall x, x' \in \mathbb{D}^r$

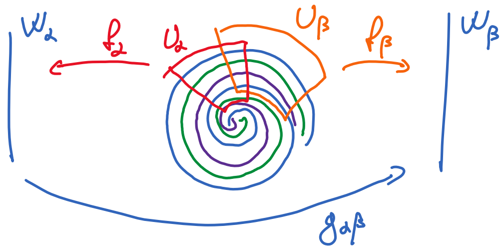
As for atlases on manifolds, one says that two foliation atlases are equivalent if their union is a foliation atlas, and equivalence classes of atlases are uniquely represented by maximal foliation atlases.

- by local submersions. $\mathcal{F}$ is (the equivalence class of) a family $\mathcal{S} = (U_\alpha, f_\alpha)$ of submersions $f_\alpha : U_\alpha \twoheadrightarrow W_\alpha \subseteq \mathbb{C}^q$ such that

  ($d_p f_\alpha$ surjective)

  $\exists\, g_{\alpha\beta} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$ s.t.

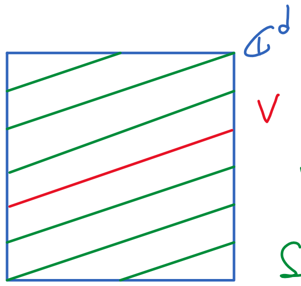
  $$f_\beta = g_{\alpha\beta} f_\alpha$$



As before, two submersion atlases are equivalent if their union is a submersion atlas, and their equivalence classes are represented by maximal atlases.
In this case, plaques are given by $f_\alpha^{-1}(c)$, and the cocycle condition ensures that they glue together into leaves.
Rem: if we replace $\mathbb{D} \subseteq \mathbb{C}$ with $(-1,1) \subseteq \mathbb{R}$, we get the real analog concept.
One can also relax the regularity of the homeomorphisms φ_p, of the transition maps $\psi_{\alpha\beta}$, or of the cocycle maps $g_{\alpha\beta}$, and get foliations of regularity $\mathcal{C}^k, \mathcal{C}^\infty, \mathcal{C}^\omega$ ($k \geq 0$, $k \geq 1$ when working with submersions).

Examples

1) On $\mathbb{C}^d$, V = linear subspace of dimension r.
Then we get a foliation $\mathcal{F}_V$ whose leaves are exactly the elements in $\mathbb{C}^d/V$



Fix $(\underbrace{v_1, .., v_r}_{\text{basis of } V}, \underbrace{w_1, .., w_q}_{\text{spans } W})$ basis of $\mathbb{C}^d$, $\mathbb{C}^d = V \oplus W$

$\forall z \in \mathbb{C}^d$, $\exists!\, (x,y) \in \mathbb{C}^r \times \mathbb{C}^q$ s.t. $z = \sum_i x_i v_i + \sum_j y_j w_j$

Set $\varphi(z) = (x,y)$, $U = \varphi^{-1}(\mathbb{D}^r \times \mathbb{D}^q)$

• local models: $\forall p \in \mathbb{C}^d$, $U_p = p+U$, $\varphi_p : U_p \to \mathbb{D}^r \times \mathbb{D}^q$, $\varphi_p(p+z) := \varphi(z)$

• transition maps $\varphi_{\alpha\beta}(x,y) = (x+\tau_x,\ y+\tau_y)$, $(\tau_x,\tau_y) = \varphi(\alpha-\beta)$.

• submersion: $f : \mathbb{C}^d \to \mathbb{C}^q$ $\quad f(z) = y = pr_2 \circ \varphi(z)$.

Let $\Lambda \subset \mathbb{C}^d$ be a lattice, and $X = \mathbb{C}^d/\Lambda$. Then $\mathcal{F}_V$ induces a foliation. Since the deck transformations (the automorphisms of the covering $\pi : \mathbb{C}^d \to \mathbb{C}^d/\Lambda$) leave $\mathcal{F}_V$ invariant, we get a foliation $\mathcal{F}$ on X.

$$\begin{tikzcd} U_\beta \arrow[dr, "\pi_\beta", "\simeq"'] & \mathbb{C}^d \arrow[r, "\varphi_\alpha"] \arrow[d, "\pi"] \arrow[dr, "f_\alpha"'] & \mathbb{D}^r \times \mathbb{D}^q \arrow[d, "pr_2"] \\ & X & \mathbb{D}^q \end{tikzcd}$$

local models: $\tilde{\varphi}_{\alpha\beta} = \varphi_\alpha \circ \pi_\beta^{-1}$

local submersions: $\tilde{f}_\beta = f \circ \pi_\beta^{-1}$

$\pi_\beta = \pi|_{U_\beta}$ $\quad U_\beta =$ connected component of $\pi^{-1}(U)$, $U \subseteq X$ well covered.

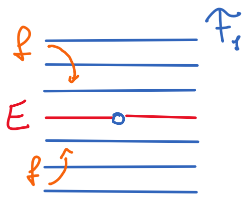
The leaves of $\mathcal{F}$ can be closed, or accumulate to submanifolds of dimension from r to d, depending on V and Λ.

2) Consider the foliation $\mathcal{F}_r$: $\{y = \text{const}\}$ on $\mathbb{C}^d \setminus \{0\}$ $(\ni (\overset{\mathbb{C}^r}{\overset{\in}{x}}, \overset{\mathbb{C}^q}{\overset{\in}{y}}))$

The map $f(x,y) = (\lambda x, \mu y)$, with $\lambda = (\lambda_1 \dots \lambda_r)$, $\mu = (\mu_1 \dots \mu_q)$ with $0 < |\lambda_j|, |\mu_j| < 1$, leaves $\mathcal{F}_r$ invariant, hence it induces a foliation $\mathcal{F}$ on the (primary) Hopf manifold $X = \dfrac{\mathbb{C}^d \setminus \{0\}}{\langle f \rangle}$

In the example $d=2$, $r=q=1$, the leaf $y=0$ gives a closed leaf isomorphic to an elliptic curve E.

Other leaves are biholomorphic to $\mathbb{C}^*$ and accumulate to E



Singular foliations of dimension 1.

Set $r=1$. We can describe the standard foliation $\mathcal{F}_{st}^{[1,d-1]}$ as the integral curves of $X_{st} = \partial_x$.

If $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{D} \times \mathbb{D}^{d-1}$ is a foliation chart, X_{st} pulls back to $X_\alpha := \varphi_\alpha^* X_{st}$ $\left(= (\varphi_\alpha^{-1})_* X_{st}\right)$, which is a nowhere vanishing vector field

Notice that any horizontal non-vanishing vector field $\tilde{X}_{st}$ would give the same integral curves ($y = \text{const}$), hence we identify vector fields X_α and $\tilde{X}_\alpha$ if $\exists u_\alpha \in \mathcal{O}^*(U_\alpha)$ s.t. $\tilde{X}_\alpha = u_\alpha X_\alpha$.

If (U_α, X_α) and (U_β, X_β) are nowhere vanishing vector fields defined on intersecting open sets U_α, U_β, then the local foliations glue together to a foliation on $U_\alpha \cup U_\beta$ as long as $\exists u_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ s.t. $X_\beta = u_{\alpha\beta} X_\alpha$.

By dropping the nowhere vanishing condition, we get singular foliations.

Def: A singular foliation $\mathcal{F}$ of dimension 1 (or by curves) on X is given by a collection $\{(U_\alpha, X_\alpha)\}$ of vector fields X_α such that $\forall \alpha, \beta$ $\exists u_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ s.t. $X_\beta = u_{\alpha\beta} X_\alpha$.

The set $\operatorname{Sing}(\mathcal{F}) := \{ p \in X \mid \exists \alpha, X_\alpha(p) = 0 \}$ is an analytic subset of X, called the singular locus of $\mathcal{F}$.

We will also assume that X_α are not identically vanishing $\iff \operatorname{Sing}(\mathcal{F})$ is a proper analytic subvariety of X (assume X connected)

Rem: $\mathcal{F}$ defines a regular foliation $\mathcal{F}_{reg}$ on $X_{reg} = X \setminus \operatorname{Sing}(\mathcal{F})$. Leaves and plaques of $\mathcal{F}$ are the ones of $\mathcal{F}_{reg}$.

Suppose now that $\operatorname{codim}\big(\overbrace{\operatorname{Sing}(\mathcal{F}) \cap U_\alpha}^{\Sigma_\alpha}\big) = 1$:

$$X_\alpha = \phi_\alpha^m \cdot \hat{X}_\alpha \quad , \text{ where } \quad \Sigma_\alpha = \{\phi_\alpha = 0\}, \; m \in \mathbb{N}^*, \; \operatorname{codim}(\operatorname{sing} \hat{X}_\alpha) \geq 2.$$

In particular, $\mathcal{F}_\alpha := \mathcal{F}\big|_{U_\alpha}$ extends to $\hat{\mathcal{F}}_\alpha$ foliation with $\operatorname{codim} \operatorname{sing}(\hat{\mathcal{F}}_\alpha) \geq 2$, called the saturation of $\mathcal{F}_\alpha$.

If (U_β, X_β) is another chart, $X_\beta = g_{\alpha\beta} X_\alpha$, and $X_\beta = \phi_\beta^m \hat{X}_\beta$ with $\operatorname{codim}(\operatorname{sing} \hat{X}_\beta) \geq 2$.

Here $m_\alpha = m_\beta = m$ because the vanishing orders coincide on $U_\alpha \cap U_\beta$.

We deduce that the saturation process globalises.

Rem: Suppose X is a manifold of dimension d, Σ is an analytic subset of codimension ≥ 2. Let $\mathcal{F}_0$ be a regular foliation on $X \setminus \Sigma$. Let $p \in \Sigma$, $U \ni p$ open neighborhood in X, and cover $U \setminus \Sigma$ by charts (U_α, χ_α). Up to shrinking U if necessary, we may assume that U is covered by a single chart of X, so that we may suppose $U \subseteq \mathbb{C}^d$.
Each χ_α can be written as $\chi_\alpha = f_\alpha^j \partial_{x^j}$ $(x^1, \dots, x^d)$ coordinates.
We may assume (up to changing coordinates linearly) that $f_\alpha^d \not\equiv 0$ $\forall \alpha$.
Then f_α^j / f_α^d are meromorphic functions defined on U_α. Moreover, $\chi_\beta = u_{\alpha\beta} \chi_\alpha$ for some $u_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, and $\frac{f_\alpha^j}{f_\alpha^d} \equiv \frac{f_\beta^j}{f_\beta^d}$ on $U_\alpha \cap U_\beta$ $(\forall j = 1 \dots d-1)$.
We deduce that $\left\{ \frac{f_\alpha^j}{f_\alpha^d}, \alpha \right\}$ glue together to meromorphic functions $\frac{f^j}{f^d} \in \mathcal{M}(U \setminus \Sigma)$

Since $\operatorname{codim} \Sigma \geq 2$, by LEVI'S EXTENSION THEOREM ([GRIFFITHS-HARRIS, p.396]), these meromorphic maps extend to U. (holomorphic, possibly vanishing)
But then the data (U_α, χ_α) is induced by $(U, \chi := f^1 \partial_{x_1} + \dots + f^d \partial_{x_d})$.
In other terms, we can define saturated singular holomorphic foliations (of dimension 1) on X as $\mathcal{F} = (\mathcal{F}_{reg}, \Sigma)$, where $\operatorname{codim} \Sigma \geq 2$ and $\mathcal{F}_{reg}$ is a regular foliation on $X \setminus \Sigma$.

Ex: Consider $\chi = \partial_x + e^{\frac{1}{x}} \partial_y$. This vector field defines a regular foliation $\mathcal{F}_0$ on $\mathbb{C}^2 \setminus \Sigma$ $\quad \Sigma = \{x = 0\}$.
$\mathcal{F}_0$ does not extend to a singular foliation $\mathcal{F}$ on $\mathbb{C}^2$, due to the fact that $x \mapsto e^{\frac{1}{x}}$ has an essential singularity at 0.
If $\mathcal{F}_0$ extends, $\exists\, U$ open neighborhood of $p = (0,0)$, ξ vector field on U, and $u \in \mathcal{O}^*(U)$ such that $\xi = u \chi$. But on U χ takes the value $\partial_x + c\, \partial_y$ ∞-many times $\forall c$ (but at most 1 value).
This is a contradiction, since $\xi = f(x,y)\partial_x + g(x,y)\partial_y$, with f, g holomorphic.

Singular foliations of codimension 1.

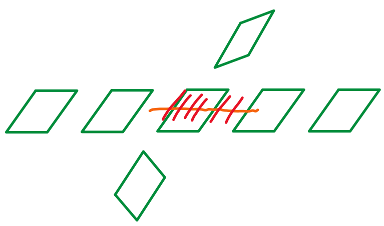
Here we try to replace vector fields with 1-forms:
when $q=1$, we can describe $\mathbb{D}^d = \bigsqcup_{y\in\mathbb{C}} \mathbb{D}^{d-1}\times\{c\}$ as the family of hypersurfaces tangent to $\ker\omega$, where $\omega = dy$

Since $\ker\omega$ stays invariant by multiplication by non-vanishing function, we get:

Def: A singular foliation $\mathcal{F}$ of codimension 1 (or by hypersurfaces) on X is given by a collection $\{(U_\alpha, \omega_\alpha)\}$ of integrable 1-forms ω_α such that $\forall \alpha,\beta\ \exists\, v_{\alpha\beta}\in\mathcal{O}^*(U_\alpha\cap U_\beta)$ s.t. $\omega_\beta = v_{\alpha\beta}\,\omega_\alpha$.
The set $\mathrm{Sing}(\mathcal{F}) := \{p\in X \mid \exists\alpha,\ \omega_\alpha(p)=0\}$ is an analytic subset of X, called the singular locus of $\mathcal{F}$.

Problem: not all distribution of hyperplanes are integrable!
Thm (Frobenius): A distribution H of planes is integrable if and only if it is involutive: $\forall\, X_1, X_2\in H$, $[X_1,X_2]\in H$.
When $H=\ker\omega$ is a distribution of hyperplanes, it is involutive $\iff$ $\omega\wedge d\omega = 0$.

Ex: [Lee2, Ex.19.2d] in $\mathbb{C}^3$, $\omega = dz + x\,dy$ $\quad$ $\omega\wedge d\omega = dz\wedge dx\wedge dy \neq 0$.
$\ker\omega$ is nowhere integrable:
An integral surface S at $(0,0,0)$ should have to contain $\{0\}\times\mathbb{D}_\varepsilon\times\{0\}$ for $\varepsilon\ll 1$ because $\partial_y|_{x=0}\in\ker\omega$, and $\mathbb{D}_\delta\times\{y\}\times\{0\}$ for $|y|<\varepsilon$, $\delta\ll 1$, because $\partial_x\in\ker\omega$.



Hence we must have $S=\{z=0\}$ locally at the origin, a contradiction.

Rem: If $d=2$, $q=r=1$, the condition $\underbrace{\omega\wedge d\omega}_{3\text{-form}}=0$ is trivially satisfied

In this case, one can define locally singular foliations equivalently by a vector field χ or by a 1-form ω.
One goes from one to another by: $\chi = f\partial_x + g\partial_y \rightsquigarrow \omega = f dy - g dx$.

Tangent and (co)-normal sheaves

Let $\mathcal{F}$ be a regular foliation of dimension r (codimension q) on a manifold X of dimension $d = r+q$.
The kernel $\ker df_\alpha$ of the local submersions f_α glue together to define a subbundle $T_{\mathcal{F}}$ of TX, called the tangent bundle of $\mathcal{F}$.
The quotient $N_{\mathcal{F}} := {}^{TX}/_{T_{\mathcal{F}}}$ is called the normal bundle of $\mathcal{F}$, and its dual $N^*_{\mathcal{F}}$, called the conormal bundle of $\mathcal{F}$, is a subbundle of Ω^1_X, whose local sections consist of the 1-forms vanishing along the leaves of $\mathcal{F}$.

Let now $\mathcal{F}_0$ be a regular foliation on $X \setminus \Sigma$, where Σ is an analytic subset of X.
By following the case of dimension and codimension 1, when $\operatorname{codim} \Sigma \geq 2$, we can define a saturated singular foliation $\mathcal{F}$ on X as a regular foliation $\mathcal{F}_0$ on $X \setminus \Sigma$.
Rem: in doing so, it is not clear how to define the singular locus $\operatorname{sing} \mathcal{F} \subseteq \Sigma$.
We would like a concept that does not depend on the choice of Σ. Or equivalently, we would like to pick Σ minimal, by extending $\mathcal{F}_0$ wherever possible.
The tangent and conormal bundles of $\mathcal{F}_0$ do not extend to X as bundles, but as subsheaves (sub $\mathcal{O}_X$-modules) of T_X and Ω^1_X respectively.

These subsheaves are coherent of rank r and q respectively.
Moreover, we have $T_F^{\perp} = N_F^*$, $(N_F^*)^{\perp} = T_F$ (saturation), and in particular T_F and N_F^* are reflexive.
By FROBENIUS' theorem, T_F must also be involutive.
The singularities of $\mathcal{F}$ are the points $x \in X$ where T_X/T_F, or equivalently Ω^1_X/N_F^*, are not locally free. ($\Rightarrow$ codim Sing $\mathcal{F} \geq 2$)
Notice that Sing(T_F) and Sing(N_F^*) have codim. ≥ 3, being reflexive [FRIEDMAN].
Notice also that if $d=2$, T_F and N_F^* are automatically locally free.
Rem: this approach works also for X a normal variety, with $\Sigma \supseteq$ Sing X.

First integrals and separatrices

Let $\mathcal{F}$ be a singular foliation on X. A first integral for $\mathcal{F}$ is a non-constant function s such that the leaves of $\mathcal{F}$ are contained in the level hypersurfaces of s ($s=$ const).
Ex: If $\mathcal{F}$ is defined by a vector field χ, the condition is: $\chi(s)=0$
If $\mathcal{F}$ is defined by a 1-form ω, the condition is: $\omega \wedge ds = 0$
Not all foliations admit first integrals. The ones that do are called integrable
Ex: $\chi = x\partial_x + \alpha y \partial_y$ $\alpha \in \mathbb{C}^*$ take $s = y x^{-\alpha}$. Then

$$\chi(s) = x(-\alpha x^{-\alpha-1}y) + \alpha y x^{-\alpha} = 0$$

Hence s defines a first integral. This is a meromorphic first integral as long as $\alpha \in \mathbb{Z}$. If not, we get a "multivalued" first integral, that still helps to describe the leaves $y = cx^{\alpha}$, which are transcendental whenever $\alpha \notin \mathbb{Q}$ and $c \neq 0$.
A related concept is the one of (complex) separatrix.

Def: Let $\mathcal{F}$ be a singular foliation. A separatrix at a point $p \in X$ is a (germ of) analytic subvariety C of dimension $r = \dim \mathcal{F}$, with $p \in C$, and such that $C \setminus \operatorname{Sing} \mathcal{F}$ is a union of leaves of $\mathcal{F}$. (Often C is irreducible at p)

Ex: the foliations above have separatrices $\{x=0\}$ and $\{y=0\}$.

If $\alpha \in \mathbb{N}^*$, then $y = c x^\alpha$ are also separatrices $\forall c \in \mathbb{C}$

More generally, if $\alpha = \frac{p}{q} \in \mathbb{Q}_{>0}$, then $\{y^q = c x^p\}$ are separatrices $\forall c \in \mathbb{C}$

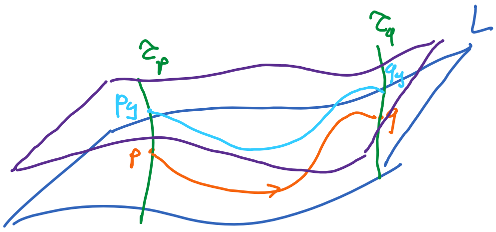
If $\alpha \notin \mathbb{Q}_{>0}$, then $x=0$ and $y=0$ are the only separatrices.

Holonomy

Let $\mathcal{F}$ be a regular foliation on X, L a leaf of $\mathcal{F}$, and $p \in L$ a point.

Pick a foliation chart (U, φ) of $\mathcal{F}$ with $U \ni p$.

A (local) transverse section τ_p at p is a (germ of) q-dimensional manifold that is transverse to $\mathcal{F}$.



It can be picked as $\varphi^{-1}(x = x_0)$ where $\varphi(p) = (x_0, y_0)$

Rem: in terms of local submersions, transversals are local sections of the submersion $f_\alpha : U_\alpha \to (\mathbb{C}^q, y_0)$

Take now another point $q \in U \cap L$, and let τ_q be a transverse section at q.

Any plaque L_y with y close to y_0 intersects τ_p in a unique point p_y.

Up to taking y close to y_0, L_y also intersects τ_q at a unique point q_y.

We set $h_{\tau_p}^{\tau_q}(p_y) := q_y$

Let now γ be a path in L joining two points p and q in L.

By compactness, we can find $t_0 = 0 < t_1 < \dots < t_N = 1$ s.t. $\gamma|_{[t_j, t_{j+1}]}$ is contained in a foliation chart U_j.

Pick transverse sections $\tau_p = \tau_0, \tau_1, \dots \tau_N = \tau_q$ at $\gamma(t_0), \dots, \gamma(t_N)$. We consider
$$h_{\tau_{N-1}}^{\tau_N} \circ \dots \circ h_{\tau_1}^{\tau_2} \circ h_{\tau_0}^{\tau_1},$$
which does not depend on the choice of t_j, nor of τ_j, $j = 1 \dots N-1$. We denote it by $h_{\tau_p, \gamma}^{\tau_q} : (\tau_p, p) \to (\tau_q, q)$.

Notice that $h^{\tau_q}_{\tau_p,\gamma}$ depends only on the homotopy class $[\gamma]$ of γ in L (relative to the extremes p,q). To see this, cover $[0,1]^2$ with finitely many small squares Q_j so that $\Gamma(Q_j)$ are in a foliation chart.

Finally, if we replace τ_p by τ_p' and τ_q by τ_q', we get:

$$h^{\tau_q'}_{\tau_p',[\gamma]} = h^{\tau_q'}_{\tau_q} \circ h^{\tau_q}_{\tau_p,[\gamma]} \circ \underbrace{h^{\tau_p}_{\tau_p'}}_{= \left(h^{\tau_p'}_{\tau_p}\right)^{-1}} .$$

In particular, if γ is a loop at $p=q$, and $\tau_p = \tau_q$, then $h^{\tau_p}_{\tau_p[\gamma]}$ determines a unique conjugacy class, denoted h_γ or $h_{[\gamma]}$, of (invertible) germ at $(\mathbb{C}^q,0)$, and called the holonomy (with respect to the loop γ).

This defines a group homomorphism $h: \pi_1(L,p) \to \mathrm{Aut}(\mathbb{C}^q,0)/_{\simeq}$.

This construction applies to any leaf L of the foliation $\mathcal{F}$. We will focus mainly to the case of loops around the singularity of a separatrix.

Ex: $X = x\partial_x + 2y\partial_y$, $C = \{y=0\}$ separatrix, $\gamma = (e^{2\pi i t}, 0)$.

Compute h_γ

It suffices to split γ at $\frac{1}{2}$ (to have determinations of the logarithm)

Take transverse sections $\tau_0 : x=1$ and $\tau_1 : x=-1$

at $t=0$, we have the point $(1, y_0)$.

At $t=\frac{1}{2}$, we have $L_{y_0} = \{ y = y_0 x^2 = y_0 e^{2\log_+ x} \}$, where $\log_+$ is the determination of log with image in $\{ \mathrm{Im}\, z \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \}$.

We get $h_0^{\frac{1}{2}}(y_0) = (-1, y_0 e^{i\pi 2})$.

Similarly, by getting the determination $\log_-$ of log whose image is in $\{ \mathrm{Im}\, z \in (\frac{\pi}{2}, \frac{5\pi}{2}) \}$, we finally get $h_\gamma(y_0) = y_0 e^{2\pi i 2}$.

One can organise holonomies into the holonomy groupoid:
the objects are the points in $X \setminus \mathrm{Sing}\,\mathcal{F}$, and the morphisms are holonomies with respect to (homotopy classes) of paths joining points inside a common leaf. In order to do so, one can fix a field of transverse sections...
This groupoid differs from the monodromy groupoid, where the morphisms are the homotopy classes of paths tangent to $\mathcal{F}$ (i.e., inside leaves).